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1-Dimensional Advection-Diffusion Finite Difference Model Due to a Flow under Propagating Solitary Wave

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Abstract. Advection–diffusion phenomena has been commonly observed in coastal areas. Our aim is to investigate the solitary wave effect to advection–diffusion of a substance in a near shore shallow water with an open channel. Considering that the model is an open channel, one dimensional approach is applied. The model solution is obtained by combining Korteweg – De Vries (KdV) equation with advection – diffusion equation. The KdV equation is numerically solved using semi-implicit Crank-Nicolson scheme while the advection – diffusion equation is numerically solved by using an upwind – Forward Time Central Space (FTCS) scheme. The results show that there is a rapid dispersion of the substance when the solitary wave comes into contact with the source

Keywords: advection-diffusion, solitary wave, finite difference method.

1. Introduction

Transport and dispersion phenomena which are described by advection-diffusion equation have been common problems and observed in a wide range of industrial and engineering applications [1]. Somehow, it is important to investigate the advection-diffusion process of a contaminant especially near shore. Many contaminants from industrial activity end up at the coastal zone. This advection-diffusion phenomena also plays an important role in keeping the balance of the coastal environment such as in heat exchange and salinity transport phenomena.

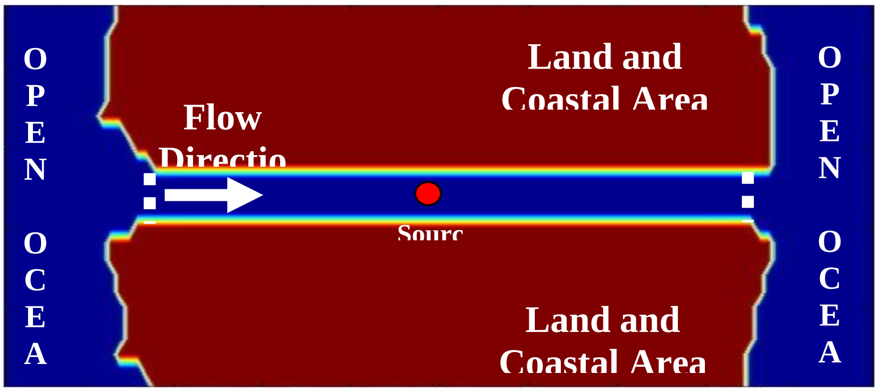


Fig. 1: Sketch of the open channel

In this paper, our aim is to investigate advection-diffusion process of a substance near shore due to a flow under a propagating solitary wave. We consider a simple case of advection-diffusion phenomena in 1 dimensional domain such as in an open channel where the solitary wave enters from the open ocean.

The model is obtained by solving 1 dimensional Korteweg De-Vries (KdV) and 1 dimensional advection-diffusion equation with the help of finite difference method. In this study, the solitary wave is used because it's been proven by Munk [2] that a solitary wave theory gives a better approximation for near shore wave than linear wave theory.

One dimensional KdV equation for an even bottom read as [3]

$$\frac{\partial \eta}{\partial t}+C\frac{\partial \eta}{\partial x}+\mu\eta\frac{\partial \eta}{\partial x}+\delta\frac{\partial^3 \eta}{\partial x^3}=0 \tag{1}$$

where $C=\sqrt{gh}$, $\mu=\frac{3}{2}\frac{C}{h}$ and $\delta=\frac{1}{6}Ch^2$ with h is the water depth, g is the gravitational acceleration and η is the surface elevation. One dimensional advection-diffusion equation can be expressed as

$$\frac{\partial F}{\partial t}=-u\frac{\partial F}{\partial x}+A_D\frac{\partial^2 F}{\partial x^2} \tag{2}$$

where F describes the concentration of a substance, $u=u(x,t)$ represents velocity of a fluid flow as a function of space and time and A_D is the diffusivity constant.

Equation (1) and (2) will be solved by using finite difference to produce 1 dimensional advection-diffusion numerical model.

2. Numerical Method

2.1. Solitary Wave model From KdV Equation

The KdV equation (1) will be solved by using Crank-Nicolson scheme. By following [4], the final form of equation (1) after the discretization is read as

$$b_2\eta_{i-2}^{n+1}+b_1\eta_{i-1}^{n+1}+a_0\eta_i^{n+1}+a_1\eta_{i+1}^{n+1}+a_2\eta_{i+2}^{n+1}=d \tag{3}$$

where the left side coefficients are

$$a_0=\frac{-2\Delta t}{\Delta x}\left(C+\mu\eta_i^n\right)+\frac{6\Delta t}{\Delta x^3}\delta$$

$$a_1=1+\frac{\Delta t}{\Delta x}\left(C+\mu\eta_{i+1}^n\right)-\frac{4\Delta t}{\Delta x^3}\delta$$

$$b_1=-1+\frac{\Delta t}{\Delta x}\left(C+\mu\eta_{i-1}^n\right)-\frac{4\Delta t}{\Delta x^3}\delta$$

$$a_2=\mathrm{b}_2=\frac{\Delta t}{\Delta x^3}\delta$$

and the right side coefficient is described as

$$d=\eta_{i+1}^n-\eta_{i-1}^n-\frac{C\Delta t}{\Delta x}\left(\eta_{i+1}^n-2\eta_i^n+\eta_{i-1}^n\right)-\frac{\Delta t}{\Delta x^3}\delta\left(\eta_{i+2}^n-4\eta_{i+1}^n+6\eta_i^n-4\eta_{i-1}^n+\eta_{i-2}^n\right)$$

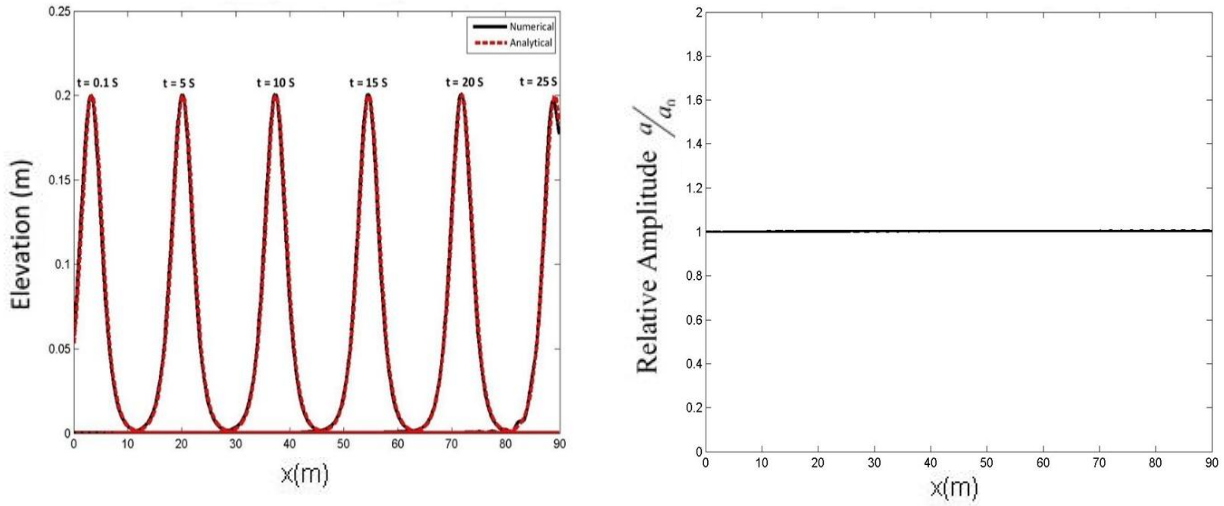
Equation (3) is then solved by Gaussian Elimination method considering that the model domain is small enough so it doesn't need much computational time. Initial condition for (1) is

$$\eta(x,0)=a\,\mathrm{sec}h^2\left(k\left(x-x_0\right)\right),\text{ where } k=\sqrt{\frac{3a}{4h^3}}$$

with x_0 is the initial peak position and a is the wave amplitude where the wave propagates from left to right. For the right boundary condition, an extrapolation boundary is applied to make the wave travels without reflection at the boundary.

2.2. Solitary Wave Test

For the solitary wave test, consider a wave flume with 90 m length and with a constant depth 1 m. We choose wave amplitude 0.2 m, $\Delta x=0.15$ m and $\Delta t=0.01$ s with wave peak position (x_0) at 3 m from left boundary with simulation time 25 s. The comparison between numerical and analytical is provided in Fig. 2 (a) where both of them are almost perfectly coincided along its propagation.



(a) (b)

Fig. 2: Comparison between numerical model (solid line) and analytical solution (dashed line) for various t (a) and relative amplitude as the wave travels (b)

It can be seen from Fig. 2 (b) that there is almost no increase or decrease in the amplitude as the wave propagates. With $a_0 = 0.2$ m where a is the numerical amplitude which is fluctuating along the propagation. The significant error occurs only at t = 25 s, see Fig. 2 (a) when the wave reaches the right boundary. But overall, the numerical model is in good agreement with the analytical solution and the right side extrapolation boundary gives a good result without any reflection.

2.3. Coupled KdV and Advection-Diffusion Model

An upwind discretization for the advection term according to [5] and FTCS discretization for the diffusive term in equation (2) is

$$F_i^{n+1} = F_i^n - 0.5(\alpha - |\alpha|)(F_{i+1}^n - F_i^n) - 0.5(\alpha + |\alpha|)(F_i^n - F_{i-1}^n) + \beta(F_{i+1}^n - 2F_i^n + F_{i-1}^n) \quad (4)$$

where $\alpha = \frac{\Delta t}{2\Delta x}(u_i^n + u_{i-1}^n)$ and $\beta = A_D \frac{\Delta t}{\Delta x^2}$ with u is obtained from the solitary wave model by using the linear wave theory relation which is written as $u(x,t) = \eta(x,t)\sqrt{g/h}$.

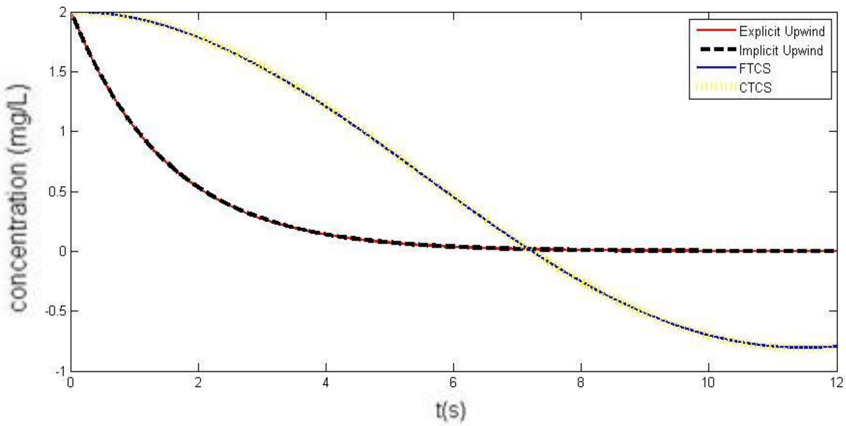


Fig. 3: Plot of concentration against time at point x = 29.85 for explicit upwind, implicit upwind, FTCS, and CTCS method

We also compare the method above with another method such as implicit upwind, FTCS, and CTCS for a pure advection process with uniform velocity 0.1 m/s which starts at t = 0 (Fig. 3). The result shows that the implicit and explicit upwind gives a better approximation than the FTCS and CTCS method. It is seen that both FTCS and CTCS methods resulting a negative concentration which is impossible to happen. A time lag is also exists in FTCS and CTCS scheme at early simulation time where the concentration is still 2 mg/L although the flow is already exist (the flow is already exist at t = 0 as mentioned before).

Although the implicit upwind method is really stable [5], we choose the explicit upwind method rather than implicit upwind method because the explicit method produces a shorter computational time. As long as the model parameter satisfy the stability criterion, the result will be almost perfectly coincided with the implicit method (see Fig. 3).

The coupled model uses the same wave set up as the first solitary wave test with same space and time resolution (Δx , Δt) but with length of channel 75 m and simulation time 12 s.

There are two scenarios simulated by this coupled model. The first scenario considers only the advection process and the second scenario considers both of advection and diffusion process with $A_D = 2\times10^{-3}$ m^2/s (we take a small diffusion constant to keep the advection process visible). Both scenario use an initial condition $F_0 = 2$ mg/L at $x = 29.85$ with insulated boundary conditions at both sides. We assume that there is no dispersion to the open ocean at the boundary.

3. Results

For the time series plot, we show only the source point time series at x = 29.85 m. From the first scenario, it can be seen from Fig. 4 (a) that a rapid decrease happens at a time interval between 5 s – 8 s. It is confirmed by the time series plot (see Fig. 4 (b)) that the rapid decrease begins at around t = 5 s to t = 8 s when the flow reaches the source point. This rapid decrease at the source point is also indicating that a rapid dispersion occurs (because the concentration is conserved). It can also be seen that before the flow reaches the source point (before t = 5 s), there is no decrease in concentration at this point.

After t = 8 s, the concentration doesn't decrease anymore although the flow still exists at the source point (Fig. 4 b). It happens because the concentration has already dispersed rapidly to the right side at interval 5 s to 8 s. It makes the concentration at the source point is almost 0 mg/L at t = 8 s. It can be seen from Fig. 4 (a) that the peak position has moved into a point between 30 m – 30.5 m with the concentration at the source position is almost 0 mg/L at t = 8 s (black solid line).

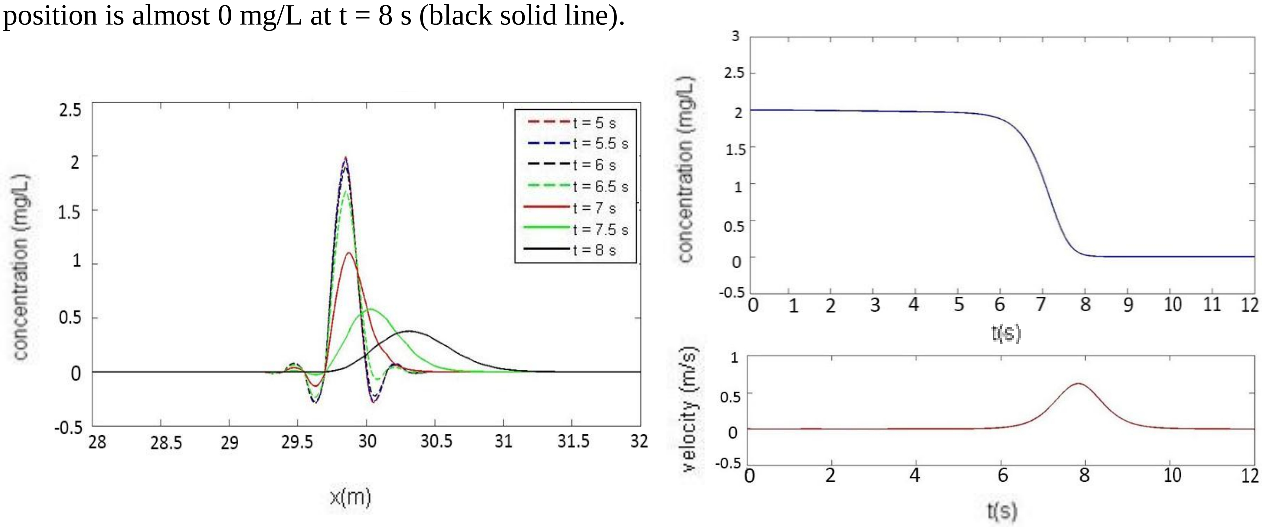
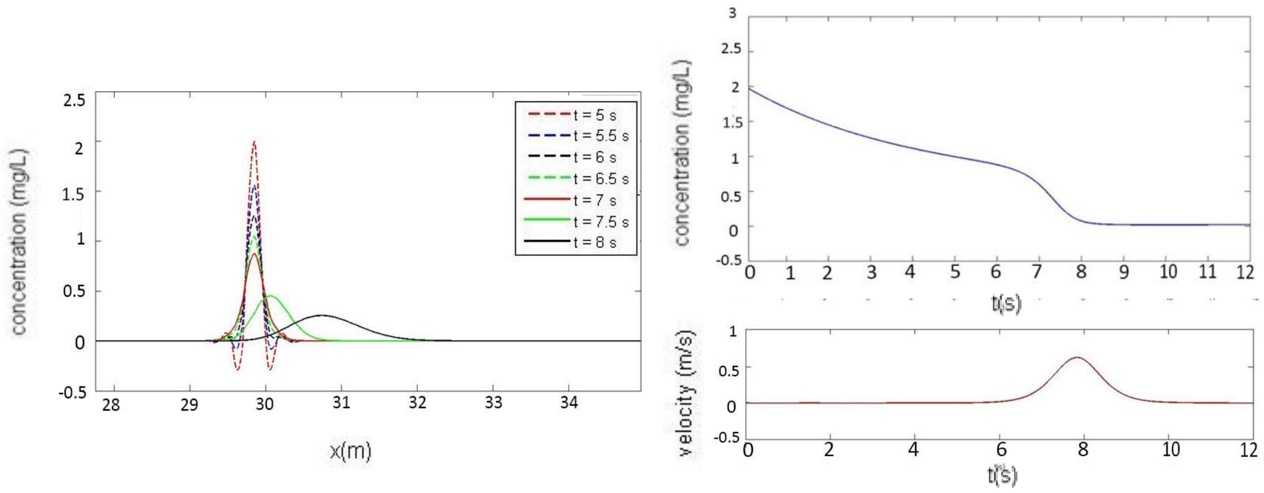


Fig. 4: Plot of concentrat (a) inst length of channel for various t (a) and plot of conce (b) (upper) and velocity (lower) against time at point x = 29.85 (b) for the 1st scenario

For the second scenario, it can be seen from Fig. 5 (a) that before t = 7 s the diffusion is more dominant than the advection. Although if we see carefully at t = 7 s, a small advection process has already begun. It is indicated by the widening of the solid red line curve to the right at its lower region. It is also confirmed by Fig. 5 (b) where the flow begins to reach the source point at around t = 6 s. It means the advection process is starting to occur at t > 6 s. But at those interval, the flow velocity is still small enough so the diffusion phenomena is still dominating.

After t > 6.5 s, the advection process dominates the dispersion which almost has the same trend as the first scenario, see Fig. 5 (b) and Fig. 4 (b). And we can also see that after t = 8 s the concentration decrement is approaching zero although the flow still exists. It is caused by the similar reason as the first scenario that at t > 8 s, the concentration at x = 29.85 m is almost 0 mg/L which is indicated by solid black line in Fig. 5 (a).



(a) (b)

Fig. 5: Plot of concentration against length of channel for various t (a) and plot of concentration (upper) and velocity (lower) against time at point x = 29.85 (b) for the 2nd scenario

We have to involve the water particle movement to explain this rapid dispersion. Unlike the linear shallow water theory, the water particle movement under a propagating solitary wave doesn't make a horizontal oscillating translational movement, see [6]. It makes a motion that following the direction of the wave propagation. So it makes the contaminated water particles always move following the direction of the wave propagation and disseminate the particle of the substance to the right continuously.

4. Conclusions

One Dimensional advection-diffusion model is used to simulate how fast a substance is dispersed into its environment in an open channel caused by a flow under solitary wave. With a small diffusion coefficient, it can be seen from the model that the advection holds a big role in dispersing the substance rapidly. The water particle movements under a propagating solitary wave is a key in explaining the rapid dispersion of a substance.

5. Acknowledgements

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