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Use of Hankel Operatorsfor Identifying Effective Empirical Orthogonal Functions in Climatology

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Abstract. This paper proposes a new method to use the techniques commonly used in system engineering for identifying the effective states in the input output behavior of the system, as the basis to identify the number of Empirical Orthogonal Functions that capture most of the variance for the purpose of reducing dimension of the observed field and identifying the climate's principal modes of variation empirically.

Keywords: EOF analysis, Climate dynamics, Statistical analysis, Model order reduction

1. Introduction

Understanding the earth's climate is currently one of the most important global concerns. Paleoclimate research has the potential to provide unique contribution to the understanding of the climate system and our ability to predict future climate change [1].The analysis of the long records of currently available data is one the most important tasks for the climate scientists.This is not only evident in paleoclimate research but also anywhere that there is a need for analysis of large datasets, including real systems as well as quasi-realistic models such as atmospheric or Oceanic General Circulation Models (GCM) [2]. In this field, usually analysing the entire dataset gathered from observation or simulations, is not only computationally expensive but also may not be useful to those who aim to know the dynamics controlling the behaviour, development and relationships in the systems. One of the commonly used techniques for data analysis is the use of Empirical Orthogonal Functions (EOFs) that aims to find new set of variables that capture most of the observed variance from the data through a linear combination of original variance [3],[4]. EOF's are computed in order toidentify the dynamically significant components in datasets. At the same time, they can be used to define patterns and indices objectively. EOFs were first introducedin climate and atmospheric sciences in 1950s and later became a popular technique for atmospheric scientists[5],[6],[7].

The EOFs can be used as a tool to reduce the size and dimensionality of the data. In that sense a common approach is to calculate the first EOF and determine how much of the variance is explained through the first EOF and use that to analyze the data. Even though in some cases the second and third EOF are also calculated to collectively cover a greater amount of explained variance, there is no coherent methodology to specify how much of the variance needs to be covered, or to what extend the amount of explained variance represents the states that contribute to the behavior of the system. On the other hand, the notion of reducing the dimensionality and model reduction is nothing new in the growing scientific community. Using balanced truncation and Hankel singular values is a common approach for model dimensionality reduction [8].Hankel singular values characterize the contribution of each state to the input-output map of the system. Using this method we can find the error for a reduced order model from the singular values of the Hankel matrix without calculating the reduced order model. Also, it is possible to obtain a bound on the norm error measurement between the impulse response of the original and reduced order model[9], [10],[11],[12].

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In this paper we propose a method to specifically identify how many of the EOFs need to be selected so that the behaviour and responses of the reduced order dataset would be similar to the original one, using model reduction techniques.

2. Methodology

In this study, the bases for understanding the EOFs and calculation methods are explained, and later a method is proposed to combine the model reduction techniques on selecting EOF modes to that are most effective in characterizing the datasets.

2.1. Model Reduction

Balanced Truncation is one of the most common and basic methods for model reduction. Therefore, in order to perform model reduction on any system one should know the balanced truncation method as the preliminary tool model reduction. In this section we explain the basics concepts of balanced truncation. Consider a higher order controllable and observable system $G(s)$. $G(s)$ is we can rewrite the system equations using the following equations:

$$\begin{aligned} \dot{\boldsymbol{x}}(\boldsymbol{t}) &= \boldsymbol{A}\dot{\boldsymbol{x}}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) &= \boldsymbol{C}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{t}) \end{aligned} \tag{1}$$

In the above equations $\boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{n}}$ represents the state, $\boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{P}}$ is the input, and $\boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{q}}$ is the output. If the transfer function of the system is strictly proper, which means the input does not directly affect the output; we can remove the D term in equation (1).

The controllability and observability gramians of the system are represented by:

$$W_c = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau \tag{2}$$

$$W_o = \int_0^t e^{A\tau} CC' e^{A'\tau} d\tau \tag{3}$$

For an asymptotically stable system the gramian matrices should satisfy the Lyapunov equations:[13]

$$AW_c + W_c A' = -BB' \tag{4}$$

$$A'W_o + W_o A = -C'C \tag{5}$$

According to [13], there exists a similarity transform $\bar{x} = Px$ such that the controllability gramian $\bar{W}_c$ and the observability gramian $\bar{W}_o$ are both equal and diagonal:

$$\bar{W}_c = \bar{W}_o = \Sigma^2 = diag[\sigma_1 \ \sigma_2 \ ... \ \sigma_k] \tag{6}$$

The new system is called the balanced realization of the original system that was represented by equation (1):

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}\bar{u}^* \\ y &= \bar{C}\bar{x} \end{aligned} \tag{7}$$

*time component t is ommited

Therefore we can rewrite the Lyapunov equations as

$$\bar{A}\Sigma + \Sigma\bar{A}' = -\bar{B}\bar{B}' \tag{8}$$

$$\bar{A}'\bar{\Sigma}+\bar{\Sigma}\bar{A}=-\bar{C}'\bar{C} \tag{9}$$

Σ^2 is a diagonal matrix, and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \geq 0$ are called the hankel singular values. Furthermore, if $\sigma_m \gg \sigma_{m+1}$ it is possible to partition Σ^2 into the following

$$\Sigma^2=\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix} \tag{10}$$

$$\Sigma_1 = diag[\sigma_1\ \sigma_2\ \ldots\ \sigma_m] \tag{11}$$

$$\Sigma_2 = diag[\sigma_{m+1}\ \sigma_{m+2}\ \ldots\ \sigma_k] \tag{12}$$

Therefore, we can modify equation (7) to:

$$\begin{bmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}u$$
$$y=[\,C_1\quad C_2\,]\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} \tag{13}$$

We can remove the states that correspond to the smaller hankel singular values Σ_2; since these states have much smaller contributions to the input-output data of the system [8]. The reduced order system can be represented by:

$$\begin{aligned}\dot{x}_{red} &= A_{11}x_{red}+B_1u\\ y_{red} &= C_1x_{red}\end{aligned} \tag{14}$$

One should consider that states could be removed if they have relatively smaller hankel singular values. For example, in a system that has singular values of the order 10^{-8} and less, only the states corresponding to the hankel singular values that are much smaller than 10^{-8} can be removed. Finally, using the above procedure, the reduced order model will have the same input-output behavior as the original system or dataset.

2.2. Empirical Orthogonal Functions and SVDs

The detailed analysis and derivation of EOF is explained in many different resources by statisticians as well as climate scientists, such as[7], [14]. However as a general rule the aim of EOF is to find a linear combination of the variables in a dataset that explain the maximum variance. The simplest method to compute the EOF is based on the Singular Value Decomposition (SVD) of the data matrix [3]. In this method the data matrix $n \times m$ matrix X can be represented as:

$$X = L\Lambda R^T \tag{15}$$

In equation (15) L is a matrix of size $m \times m$ containing the left singular vectors, R is an$n \times n$ matrix and it contains the right singular vectors and Λ is the eigenvalue matrix, that is strictly positive, diagonal matrix with the same dimension as *X*. All eigenvalues are in descending order, at the same time one must consider that maximum number of nonzero eigenvalues are equal to the *min(n,m)*. The elements in diagonal matrix are defined as the singular values where the explained variance and their contributions to the total variance can be computed using the following equations:

$$\sigma_i = \sqrt{\Lambda_i} \tag{16}$$

$$\%\sigma_i = \frac{\sigma_i}{\sum_1^n \sigma_i} \tag{17}$$

In most applications in climate sciences the EOFs are columns of L and the Principal Components (PC)s are columns of R. Further, for the purpose of reducing the dimensionality one must choose the states, eigenvalues, such that together they define a reasonable amount of variance, therefore the modes of the EOFs used are be the vectors corresponding to those specific singular values that produce the required variance.

3. Discussion and Conclusions

Explaining both concepts in detail in this paper would like to conjecture that with the tools in hand for mathematical model reduction of systems there are more efficient and scientific methods for selecting the contributing modes of EOF. In the datasets one can use the EOF analysis for reducing the dimensionality keeping in mind that for those datasets where there is a possibility of converting the dataset into its equivalent state-space system we can directly use the methods explained in section 2.1. However, once the state-space modelling is not feasible or it is computationally expensive, the techniques for choosing significant modes of EOF an be enhanced with the use of Hankel operators and concept of automatic selection of the modes that significantly contribute to the characteristic of the dataset or the behaviour of the system if the system is chosen to be converted to the linearized state-space model in specific cases.

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